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On the Attitude Dynamics of Spinning Deformable Systems

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A general method for attitude dynamics investigation of spinning deformable systems is presented. The stability of given equilibrium configurations is determined by use of the Liapunov technique. Necessary and sufficient conditions are obtained in most cases. The choice of the body reference frame is considered, and the motion of various frames, such as principal axes frame, mean frame, and Tisserand's frame, is investigated.

Introduction

The stability analysis of deformable mechanical systems is an important problem in applied mechanics and, in particular, for space technology applications. Many scientific papers have been devoted to this subject during the last decade, and realistic models have been presented even for sophisticated structures.

In a first approach to the rotational dynamics for nonrigid spacecrafts, the system is supposed to be a continuum, and its dynamical behavior is described by a set of ordinary and partial differential equations. Once an equilibrium configuration is defined, the attitude stability can be discussed from considerations on the linear displacements about that state and sufficient stability conditions can be obtained by various methods.[1–3]

Further, when the deformations are expressed in terms of a linear combination of normal modes, the behavior of the system can be described by a set of ordinary differential equations. Such an approach, introduced by Buckens in his pioneering work of 1963,[4] is very useful when the modal analysis is readily available but its applications are somewhat limited by the inevitable truncation problem.

The modelization as a continuum permits consideration of systems composed of various structures such as antenna's (booms, beams)[3,5,6] solar arrays (plates, shells),[7,8] fluid vessels, or any combination of these elements. As an example, Rumiantsev considered the rotational stability of systems, including rigid bodies, elastic parts, and cavities filled with (perfect) fluids.[9]

The multibody approach, or system discretization, is more directly oriented towards simulation but is also useful for stability investigation. Various formalisms have been developed and here Abzug's work seems to be a forerunner.[10,11] Hooker and Margulies considered a system of point connected rigid bodies and obtained vector dyadic equations expressed in terms of the angular variables corresponding to the relative motion between bodies.[12] Roberson and Wittenburg obtained independently equivalent matrix equations expressed in terms of the angular velocities of the bodies.[13] In both formalisms the torques acting in the connections were included, and the number of equations could have been larger than the number of degrees of freedom.

Hooker[14] suggested a procedure to eliminate unnecessary equations, and as suggested by Likins,[15] the equations obtained following Hooker's procedure and the Roberson-Wittenburg matrix formulation can be used for complex spacecraft simulation. A large number of computer programs were written using such ideas and simulation can then be obtained by direct integration of these nonlinear differential equations: this obviously limits the application to systems composed of a small number of equivalent rigid bodies, but appears to be very useful for a preliminary design. From that time onwards, the multibody formalism has been improved to include closed-loop configurations,[16] translations in the joints, effect of deformation, and rotors in the various bodies.[17–19]

Here also it must be noted that by an appropriate extension of Pringle's work,[20] necessary and sufficient stability conditions can be obtained from considerations on the Hamiltonian, and

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an explicit form of the equation of motion is not required for that purpose.[21]

In all the forementioned cases, the equations can be derived either from Lagrange formalisms (in generalized or quasi coordinates), Euler formalism (whatever the reference point), and/or variational principles (d'Alembert or Hamilton). The choice between these formalisms has been a point of controversy for some time, but we can conclude that the difficulties encountered in each case are comparable. Further, it is clear that the results are equivalent!

In this paper, we will develop the Lagrange equations of a general deformable system as this permits us to write the dynamical equations with a limited number of definitions and parameters. Further, this formalism permits discussion of all the systems with discrete coordinates. Such systems are obtained for continua by considering modal coordinates (possibly an infinite number) or by using a spatial discretization (lumped parameters). Systems of interconnected particles and rigid bodies lead naturally to such a formulation.

We will pay particular attention to the choice of the "body-fixed" reference frame, as this choice may somewhat simplify the analysis. Various frames were used in the forementioned papers, and we will show how to derive the equations in these different frames. We will assume that no gyrostats are included. The extention to gyrostatic systems is quite simple, but would require more space without gain in understanding.

Equations of Motion

We will consider as state variables the angular velocity of a body-fixed reference frame and internal deformation variables.

The body-fixed frame will be assumed to coincide at equilibrium with a rotating reference frame spinning with nominal angular velocity. During deformation, this frame can be fixed to a particular part of the structure (for instance the experimental package) or can be defined in a more academic way (for instance Tisserand's frame, Buckens' mean frame or axes of principal moment of inertia). It will be assumed that the origin of the reference frame always coincides with the body center of mass and that all the internal variables are equal to zero at equilibrium.

The position vector of an element of mass dm will be denoted $\boldsymbol{\rho}$ during deformation and $\mathbf{x}$ in the undeformed (equilibrium) configuration. The displacement vector $\mathbf{u} = \boldsymbol{\rho} - \mathbf{x}$ will be a function of the deformation variables β with $\mathbf{u} = 0$ for $\beta = 0$. The body frame will be denoted $\{\hat{\mathbf{X}}_\alpha\}$, and the position vector $\boldsymbol{\rho}$ will be expressed in this frame as $\boldsymbol{\rho} = [\hat{\mathbf{X}}_\alpha]^T \rho$ where

$$[\hat{\mathbf{X}}_\alpha] = [\hat{\mathbf{X}}_1 \hat{\mathbf{X}}_2 \hat{\mathbf{X}}_3]^T, \qquad \rho = [\rho_1 \rho_2 \rho_3]^T$$

ρ_α being the components of $\boldsymbol{\rho}$ in the $\{\hat{\mathbf{X}}_\alpha\}$-frame. The velocity of dm (with respect to inertial space) will then be written

$$\dot{\boldsymbol{\rho}} = \mathring{\boldsymbol{\rho}} + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

where $\mathring{\boldsymbol{\rho}} = [\hat{\mathbf{X}}_\alpha]^T \dot{\rho}$ is the "relative" velocity vector and, $\boldsymbol{\omega} = [\hat{\mathbf{X}}_\alpha]^T \omega$ is the angular velocity of the $\{\hat{\mathbf{X}}_\alpha\}$-frame.

The total kinetic energy of rotation (it is supposed that the rotational motion is uncoupled from translation) can be written:

$$T = (1/2)\boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \mathbf{h} + (1/2) \int \mathring{\boldsymbol{\rho}} \cdot \mathring{\boldsymbol{\rho}}\, dm$$

where $\mathbf{J}$ is the inertia tensor, related to the inertia matrix J by $\mathbf{J} = [\hat{\mathbf{X}}_\alpha]^T J [\hat{\mathbf{X}}_\alpha]$, and $\mathbf{h}$ is the internal angular momentum given by $\mathbf{h} = \int \boldsymbol{\rho} \times \mathring{\boldsymbol{\rho}}\, dm = [\hat{\mathbf{X}}_\alpha]^T h$.

The matrix expression of T is then:

$$T = (1/2)\omega^T J \omega + \omega^T h + (1/2) \int \dot{\rho}^T \dot{\rho}\, dm$$

As, from our assumptions, the matrices $\dot{\rho}$, J, and h are not explicit functions of ω, the Lagrange equations corresponding to the quasi-coordinates ω are simply

$$(d/dt)(\partial T/\partial \omega) + \tilde{\omega}(\partial T/\partial \omega) = L$$

where L is the matrix of the components of the resultant of external torque $\mathbf{L}$ applied to the system expressed in the $\{\hat{\mathbf{X}}_\alpha\}$-frame.

$\tilde{\omega}$ is the skew-symmetric matrix defined from the elements ω_α of the angular velocity matrix as

$$\tilde{\omega} = \begin{vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{vmatrix}$$

It must be noted that this equation is the Euler-Liouville-Resal equation of the complete system expressed in the $\{\hat{\mathbf{X}}_\alpha\}$-frame, as

$$\partial T/\partial \omega = J\omega + h = H$$

where H is related to total angular momentum by $\mathbf{H} = [\hat{\mathbf{X}}_\alpha]^T H$.

These equations suffice to describe the behavior of the system when J and h are known functions of time or when the deformations are known. Otherwise, the deformation equations are given by

$$(d/dt)(\partial T/\partial \dot{\beta}) - (\partial T/\partial \beta) = N_\beta$$

where N_β is the generalized force matrix. It should be noted that as there are no gyrostats in the system, the internal angular velocity is a linear function of the velocities $\dot{\beta}$, and the integral appearing in the expression of the kinetic energy is quadratic in $\dot{\beta}$. Nevertheless, the full derivation of the Lagrange equations in the deformable variables presents some difficulties and requires the introduction of a large number of definitions.

As, generally, the dynamical equations will be used only to simulate the behavior of the system for small perturbations about an equilibrium state and to investigate the stability of that state and as this stability is essentially determined from the linearized equations, we will make use only of their linear form. To obtain exact linearized equations, we need a complete quadratic expression of the kinetic energy (about the equilibrium state). At equilibrium, the rotating frame must coincide with the principal axes of the undeformed system. The axis of rotation will be arbitrarily denoted 3-axis and at equilibrium all the variables will be equal to zero except ω_3 which is equal to the nominal angular velocity ω_o.

To obtain a quadratic expression of T in β, ω_1, w_2, and ω_3–ω_0, we need a linear expression in β for the first two components of h and for the products of inertia J_{13} and J_{23} as well as an expansion up to quadratic terms for h_3 and J_{33}. The linear expression of $J_{12} = J_{21}$ will be used later and the elements of J and h are then written:

$$J_{11} = I_1, \quad J_{22} = I_2, \quad J_{12} = J_{21} = \beta^T \Lambda_0,$$
$$J_{13} = J_{31} = \beta^T \Lambda_1, \quad J_{23} = J_{32} = \beta^T \Lambda_2,$$
$$J_{33} = I_3 + \beta^T \Lambda_3 + \beta^T \Pi \beta$$

where $\Pi = \Pi^T$, and

$$h = \begin{vmatrix} \zeta_1{}^T \dot{\beta} \\ \zeta_2{}^T \dot{\beta} \\ \zeta_3{}^T \dot{\beta} + \beta^T \Phi \dot{\beta} \end{vmatrix}$$

It should be noted that the matrix Π may include terms due to nonlinearity (in β) of the displacements. This shows that, even for small motion, a purely linear description of the displacement field may lead to significant errors in the dynamics of rotating systems.

The integral appearing in T is written as

$$\int \dot{\rho}^T \dot{\rho}\, dm = \dot{\beta}^T M_d \dot{\beta}$$

and the potential energy of deformation will be supposed to have the form

$$U_d = (1/2)\beta^T K_d \beta + F_o{}^T \beta$$

where F_o corresponds to generalized prestresses. The equations of motion are then

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_0\omega_2 + \omega_0\Lambda_1{}^T\dot{\beta} - \omega_0{}^2\Lambda_2{}^T\beta + \zeta_1{}^T\ddot{\beta} - \omega_0\zeta_2{}^T\dot{\beta} = L_1$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_0\omega_1 + \omega_0\Lambda_2{}^T\dot{\beta} + \omega_0{}^2\Lambda_1{}^T\beta + \zeta_2{}^T\ddot{\beta} + \omega_0\zeta_1{}^T\dot{\beta} = L_2$$

$$I_3\dot{\omega}_3 + \omega_0\Lambda_3{}^T\dot{\beta} + \zeta_3{}^T\ddot{\beta} = L_3$$

$$M_d\ddot{\beta} + \omega_0\Gamma\dot{\beta} + \omega_0{}^2\Pi_d\beta - \omega_0\Lambda_1\omega_1 - \omega_0\Lambda_2\omega_2 - \omega_0\Lambda_3(\omega_3 - \omega_0) + \zeta_1\dot{\omega}_1 + \zeta_2\dot{\omega}_2 + \zeta_3\dot{\omega}_3 = N'_\beta$$

where

$$\Gamma = \Phi^T - \Phi, \quad \Pi_d = K_d/\omega_0^2 - \Pi$$

The equilibrium conditions corresponding to the variables β are then $(1/2)\Lambda_3\omega_0^2 - F_0 = 0$, if it is assumed that $N'_\beta = 0$ for $\beta = \dot{\beta} = 0$. This formalism has been developed for a system of interconnected deformable bodies. The stability of the previous linear system can be given by the Routh-Hurwitz criteria applied to the characteristic equation. This method is rather complicated, and it is impossible to obtain closed form stability conditions. We will use the Liapunov technique and look for the attitude stability with respect to the rotating reference frame.

If the orientation of the body frame with respect to the rotating frame is described by the successive rotations θ_1, θ_2, θ_3, the angular velocity matrix is in linear approximation:

$$\omega = \begin{vmatrix} \dot{\theta}_1 - \omega_0\theta_2 \\ \dot{\theta}_2 + \omega_0\theta_1 \\ \dot{\theta}_3 + \omega_0 \end{vmatrix}$$

With this value the equation can be written under the form

$$M\ddot{z} + G\dot{z} + Kz = N$$

where

$$z^T = [\theta^T \quad \beta^T] \qquad \text{with} \qquad \theta^T = [\theta_1 \quad \theta_2 \quad \theta_3],$$

$$M = \begin{vmatrix} I_1 & 0 & 0 & \zeta_1^T \\ 0 & I_2 & 0 & \zeta_2^T \\ 0 & 0 & I_3 & \zeta_3^T \\ \zeta_1 & \zeta_2 & \zeta_3 & M_d \end{vmatrix},$$

$$G = \omega_0 \begin{vmatrix} 0 & I_3 - I_1 - I_2 & 0 & -\zeta_2^T + \Lambda_1^T \\ I_1 + I_2 - I_3 & 0 & 0 & \zeta_1^T + \Lambda_2^T \\ 0 & 0 & 0 & \Lambda_3^T \\ \zeta_2 - \Lambda_1 & -\zeta_1 - \Lambda_2 & -\Lambda_3 & \Gamma \end{vmatrix},$$

$$K = \omega_0^2 \begin{vmatrix} I_3 - I_2 & 0 & 0 & -\Lambda_2^T \\ 0 & I_3 - I_1 & 0 & \Lambda_1^T \\ 0 & 0 & 0 & 0 \\ -\Lambda_2 & \Lambda_1 & 0 & \Pi_d \end{vmatrix}$$

It must be noted that the matrices M and K are symmetric matrices and that G is a skew-symmetric matrix. Further the matrix M is positive definite (except perhaps in very particular cases where it is positive semidefinite).

Freely Spinning Systems

For freely spinning systems $N = [0 \;\; N'_\beta{}^T]^T$ and the function $V = (1/2)(\dot{z}^T M\dot{z} + z^T Kz)$ can be taken as testing function. The time derivative of V along a trajectory is simply given by

$$\dot{V} = \dot{\beta}^T N'_\beta$$

If we assume that N'_β corresponds to dissipative forces, the function $\dot{V}$ is a negative semi-definite function. Even in presence of pervasive damping, the function V is not a Liapunov function as K is not positive definite.

Freely spinning systems maintain a constant angular momentum, and a suitable testing function can be obtained by combining the function V and integrals of motion corresponding to the fact that the vector $\mathbf{H}$ is constant in inertial space as already proposed.[22] In linear approximation this corresponds to

$$H_1 + H_3\theta_2 = 0$$
$$H_2 - H_3\theta_1 = 0$$
$$H_3 = H_{30}$$

with $H_{30} = I_3\omega_0$.

These relations are:

$$I_1\dot{\theta}_1 + \omega_0(I_3 - I_1)\theta_2 + \xi_1{}^T\dot{\beta} + \omega_0\Lambda_1{}^T\beta = 0$$
$$I_2\dot{\theta}_2 + \omega_0(I_2 - I_3)\theta_1 + \xi_2{}^T\dot{\beta} + \omega_0\Lambda_2{}^T\beta = 0$$
$$I_3\dot{\theta}_3 + \xi_3{}^T\dot{\beta} + \omega_0\Lambda_3{}^T\beta = 0$$

or in matrix form

$$I\dot{\theta} + \omega_0 A\theta + Z^T\dot{\beta} + \omega_0\Lambda^T\beta = 0$$

where

$$A = \begin{vmatrix} 0 & I_3 - I_1 & 0 \\ I_2 - I_3 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad Z^T = \begin{vmatrix} \zeta_1^T \\ \zeta_2^T \\ \zeta_3^T \end{vmatrix} \quad \text{and} \quad \Lambda^T = \begin{vmatrix} \Lambda_1^T \\ \Lambda_2^T \\ \Lambda_3^T \end{vmatrix}$$

We can obtain a new testing function V' by adding

$$-(1/2)(I\dot{\theta} - \omega_0 A\theta + Z^T\dot{\beta} - \omega_0\Lambda^T\beta)^T I^{-1} \times (I\dot{\theta} + \omega_0 A\theta + Z^T\dot{\beta} + \omega_0\Lambda^T\beta) \equiv 0$$

to V. As this additional term is identically equal to zero along trajectories compatible with the constraints and having compatible initial conditions, the time derivative of the new testing function V' is equal to $\dot{V}$ along all the trajectories of interest. Further, it can be easily checked that the function V' is quadratic in the variables θ_1, θ_2, β and $\dot{\beta}$, or

$$V' = y^T K' y$$

where

$$y^T = [\theta_1 \quad \theta_2 \quad \beta^T \quad \dot{\beta}^T]$$

and

$$K' = \omega_0^2 \begin{vmatrix} \frac{I_3}{I_2}(I_3 - I_2) & 0 & -\frac{I_3}{I_2}\Lambda_2^T & 0 \\ 0 & \frac{I_3}{I_1}(I_3 - I_1) & \frac{I_3}{I_1}\Lambda_1^T & 0 \\ -\frac{I_3}{I_2}\Lambda_2 & \frac{I_3}{I_1}\Lambda_1 & \Pi_d + \Lambda I^{-1}\Lambda^T & 0 \\ 0 & 0 & 0 & \frac{M_d - ZI^{-1}Z^T}{\omega_0^2} \end{vmatrix}$$

When the matrix K' is positive definite, the system is asymptotically stable (except in very particular cases for which damping remains incomplete) in the variables θ_1, θ_2, β, $\dot{\beta}$, and by use of the equations and the constraint in H it can be concluded that the linear (and the corresponding nonlinear) system is asymptotically stable. When the matrix K' can assume negative values, the system is clearly unstable from the Cetaev theorem.

The aforementioned results imply that $I_3 > I_2$ and $I_2 > I_1$ and the system must spin about its axes of maximum moment of inertia. The choice of the transverse axis was arbitrary, and it is then normal that the stability does not directly depend on the relative value of I_1 and I_2. Other stability conditions depend on the deformation of the system. Further it can be proved that $M_d - ZI^{-1}Z^T$ is always positive, and the remaining stability conditions are functions of the internal stiffness parameters for which they provide lower bounds.

Choice of the Reference Frame

The choice of a reference frame may be imposed by the choice of internal variables and also may somewhat simplify the structure of the equations and the stability analysis. Particular reference frames will be denoted as $\{\hat{\mathbf{X}}_z{}^*\}$-frames, and star variables and parameters will be defined with respect to these particular frames.

A mean reference frame is by definition a frame such that the linear part (in the variables β) of the integral $\int \mathbf{x}^* \times \mathbf{u}^* \, dm$ is equal to zero.[23] This implies that the linear part of the internal momentum $\mathbf{h}^*$ is equal to zero and consequently $\xi_1{}^* = \xi_2{}^* = \xi_3{}^* = 0$. In the case of linear displacements and if the variables describe the normal modes of the corresponding free systems, this frame follows the "rigid" modes of the system and reduces to a Buckens' mean frame.[4]

For a principal axis frame, the internal variables are such that the products of inertia are equal to zero at any instant of time. In linear approximation, this implies that $\Lambda_o{}^*$, $\Lambda_1{}^*$, and $\Lambda_2{}^*$ be equal to zero.

This frame is used in Ref. 21, and in this paper it is noted that uncoupled "external" and "internal" stability criteria are then obtained. We showed that the external stability criteria (major axis requirement) can be obtained in whatever reference frame we choose and that the simplification occurs in the expression of the internal stability criteria.

In the case of Tisserand's frame (frame such that the internal angular momentum $\mathbf{h}^*$ is equal to zero), $\zeta_1{}^* = \zeta_2{}^* = \zeta_3{}^* = 0$ and further $\Phi^* = 0$. In all these cases, the stability criteria are given by conditions on the minor of the corresponding K'^* matrix,

$$K'^* = \omega_0{}^2 \times \begin{vmatrix} -\dfrac{I_3}{I_2}(I_3-I_2) & 0 & \dfrac{I_3}{I_2}\Lambda_2{}^{*T} & 0 \\ 0 & \dfrac{I_3}{I_1}(I_3-I_1) & \dfrac{I_3}{I_1}\Lambda_1{}^{*T} & 0 \\ -\dfrac{I_3}{I_2}\Lambda_2{}^* & \dfrac{I_3}{I_1}\Lambda_1{}^* & \Pi_d{}^*+\Lambda^* I^{-1}\Lambda^{*T} & 0 \\ 0 & 0 & 0 & \dfrac{M_d{}^*-Z^*I^{-1}Z^{*T}}{\omega_0{}^2} \end{vmatrix}$$

In most cases when deformations are described by arbitrarily chosen variables, the equations will not provide the motion of one of the previous frames. If such a motion has to be known, one must consider the motion of the two frames $\{\hat{\mathbf{X}}_\alpha\}$ and $\{\hat{\mathbf{X}}_\alpha{}^*\}$ which are kinematically related by

$$[\hat{\mathbf{X}}_\alpha] = A[\hat{\mathbf{X}}_\alpha{}^*]$$

and have a relative angular velocity ω' expressed in the $\{\hat{\mathbf{X}}_\alpha\}$-frame as $\omega' = [\hat{\mathbf{X}}_\alpha]^T\omega'$.

If the angular velocity of the $\{\hat{\mathbf{X}}_\alpha{}^*\}$-frame is written $\omega^* = [\hat{\mathbf{X}}_\alpha{}^*]^T\omega^*$, the angular velocity of the $\{\hat{\mathbf{X}}_\alpha\}$-frame will be

$$\omega = \omega^* + \omega' = [\hat{\mathbf{X}}_\alpha]^T\omega = [\hat{\mathbf{X}}_\alpha]^T(A\omega^* + \omega')$$

and

$$\omega = A\omega^* + \omega'$$

The kinetic energy of the system can be written:

$$T = (1/2)(A\omega^*+\omega')^T J(A\omega^*+\omega') + (A\omega^*+\omega')^T h + (1/2)\int \dot{u}^T\dot{u}\,dm$$

or

$$T = \omega^{*T}J^*\omega^* + \omega^{*T}h^* + (1/2)\int \dot{u}^{*T}\dot{u}^*\,dm$$

where

$$J^* = A^T J A, \quad h^* = A^T(J\omega'+h), \quad u^* = A^T\rho - x$$

For Tisserand's frame, h^* is equal to zero and the relative angular velocity ω' is then given by $\omega' = -J^{-1}h$ or in linear approximation by $\omega' = -I^{-1}Z^T\dot{\beta}$ and the relative orientation can in principle be obtained from the equations $\dot{A} = -\tilde{\omega}'A$.

Other frames are not defined in such a simple manner, and their definition can be considered as a constraint between the variables. The Lagrangian is indeed a function of more variables than the degrees of freedom and constraints must be introduced so that the solution be determined uniquely.

For mean and principal axes frames the constraints correspond, respectively, to $\xi_\alpha{}^* = 0$ $(\alpha = 1,2,3)$ and $\Lambda_\alpha{}^* = 0$ $(\alpha = 0,1,2)$ and can be written under the differential form:

$$C_o{}^T d\theta' + C^T d\beta = 0$$

where the elements of the matrix θ' are the three angles describing the orientation of $\{\hat{\mathbf{X}}_\alpha\}$ with respect to $\{\hat{\mathbf{X}}_\alpha{}^*\}$.

The Lagrange equations in the quasi-coordinates ω^* are

$$(d/dt)(\partial T/\partial\omega^*) + \tilde{\omega}^*(\partial T/\partial\omega^*) = 0$$

with

$$(\partial T/\partial\omega^*) = H^* = A^T J A\omega^* + A^T J\omega' + A^T h = J^*\omega^* + h^*$$

or

$$\dot{H}^* + \tilde{\omega}^* H^* = J^*\dot{\omega}^* + \tilde{\omega}^* J^*\omega^* + \dot{J}^*\omega^* + \dot{h}^* + \tilde{\omega}^* h^* = 0$$

The Lagrangian is an explicit function of the angles θ' (in the matrix A) and the equations in the quasi-coordinates ω' are then given by

$$(d/dt)(\partial T/\partial\omega') + \tilde{\omega}'(\partial T/\partial\omega') - (\partial T/\partial\pi) + (B^{-1})^T C_0\lambda = 0$$

where $d\pi = B\,d\theta'$, the matrix B being a function of θ' given by the relation $\omega' = B\dot{\theta}'$, and λ is the vector of Lagrange multipliers.

Noting that for every vector x and y such that

$$(\partial x/\partial\theta') = (\partial y/\partial\theta') = 0$$

one has

$$(\partial/\partial\pi)(x^T A y) = \tilde{x}Ay \quad\text{and}\quad (\partial/\partial\pi)(x^T A^T y) = -A\tilde{x}A^T y$$

the equations simply reduce to:

$$A\dot{H}^* + A\tilde{\omega}^* H^* + (B^{-1})^T C_0\lambda = 0$$

It should be noted that these equations are similar to those in ω^* except for the premultiplication by the matrix A, and the presence of generalized forces due to the constraints. The equation for the variables β are now

$$(d/dt)(\partial T/\partial\dot{\beta}) - (\partial T/\partial\beta) - N_\beta + C\lambda = 0$$

We now have a complete set of equations in the proper number of variables (θ^*, θ', β, and λ) by adjoining to the previous equation the constraint equation under the form $C_0{}^T\dot{\theta}' + C^T\dot{\beta} = 0$. Since it can be checked, the linearized system has a rather simple form because the equations are equivalent in linear approximation to the equations derived from a Lagrangian which is a function of $(\theta+\theta')$ and β.

Further, $\theta' = 0$ and $\beta = 0$ being a solution, the constraint equation can be integrated as:

$$C_0{}^T\theta' + C^T\beta = 0$$

and the equation can be written:

$$M\ddot{z} + G\dot{z} + Kz = 0$$

where

$$z = [\theta'^T\theta^{*T}\beta^T\lambda^T]^T$$

$$M = \begin{vmatrix} I & I & Z^T & 0 \\ I & I & Z^T & 0 \\ Z & Z & M_d & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad G = \begin{vmatrix} G_o & G_o & -G_a{}^T & 0 \\ G_o & G_o & -G_a{}^T & 0 \\ G_a & G_a & \Gamma & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$K = \begin{vmatrix} K_o & K_o & K_a{}^T & C_o \\ K_o & K_o & K_a{}^T & 0 \\ K_a & K_a & \Pi_d & C \\ C_o{}^T & 0 & C^T & 0 \end{vmatrix}$$

with

$$G^o = \begin{vmatrix} 0 & I_3-I_1-I_2 & 0 \\ I_1+I_2-I_3 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad K_o = \begin{vmatrix} I_3-I_2 & 0 & 0 \\ 0 & I_3-I_1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$G_a = [\zeta_2-\Lambda_1 \quad -\zeta_1-\Lambda_2 \quad -\Lambda_3]$$

$$K_a = [-\Lambda_2 \quad \Lambda_1 \quad 0]$$

An equivalent linear system can be obtained by replacing θ' by its value given by the constraint equation; i.e., $\theta' = -C_0{}^{-1}C\beta$ in the expression of the Lagrangian and by deriving the new Lagrange equations in θ^* and β.

It must be noted that this system can also be obtained from the equations in z by premultiplying the system by the matrix

$$\begin{vmatrix} 0 & E & 0 & 0 \\ -CC_0{}^{-1} & 0 & E & 0 \end{vmatrix}$$

and by replacing θ' by its value given by the constraint equations.[20] This system now has the form

$$M^*\ddot{z}^* + G^*\dot{z}^* + K^*z^* = 0$$

where

$$z^* = [\theta^{*T}\beta^T]^T$$

Using one of these methods, one obtains

$$M^* = \begin{vmatrix} I & Z^T - IC_0{}^{-1T}C^T \\ Z - CC_0{}^{-1}I & \begin{matrix} M_d + CC_0{}^{-1}IC_0{}^{-1T}C^T \\ -ZC_0{}^{-1T}C^T - CC_0{}^{-1}Z^T \end{matrix} \end{vmatrix} = \begin{vmatrix} I & Z^* \\ Z^* & M_d{}^* \end{vmatrix}$$

$$G^* = \begin{vmatrix} G_o & -G_a{}^T - G_oC_0{}^{-1T}C^T \\ G_a - CC_0{}^{-1}G_o & \begin{matrix} \Gamma + CC_0{}^{-1}G_oC_0{}^{-1T}C^T \\ -G_aC_0{}^{-1T}C^T + CC_0{}^{-1}G_a{}^T \end{matrix} \end{vmatrix} = \begin{vmatrix} G_o & -G_a{}^{*T} \\ G_a{}^* & \Gamma^* \end{vmatrix}$$

$$K^* = \begin{vmatrix} K_o & K_a^T - K_o C_0^{-1T} C^T \\ K_a - C C_0^{-1} K_o & \Pi_d + C C_0^{-1} K_o C_0^{-1T} C^T \\ & - K_a C_0^{-1T} C^T - C C_0^{-1} K_a^T \end{vmatrix} = \begin{vmatrix} K_o & K_a^{*T} \\ K_a^* & \Pi_d^* \end{vmatrix}$$

The corresponding matrix K'^* is then obtained in a straightforward manner.

For mean frames, the constraints are given by

$$\int \mathbf{x}^* \times \mathbf{u}^* \, dm = 0$$

In linear approximation, this constraint may be written as

$$I\theta' + Z^T \beta = 0$$

we just have to replace C_0 by I and C by Z in the expressions of M^*, G^*, K^*, and K'^* to obtain the linearized equation of motion and the corresponding stability criteria.

It is clear that $Z^* = Z - ZI^{-1}I = 0$, and it should also be noted that $M_d^* = M_d - ZI^{-1}Z^T$ which shows that this quantity is positive as,

$$\dot{\beta} M_d^* \dot{\beta} = \int \dot{u}^{*T} \dot{u}^* \, dm$$

The stability criteria obtained from the minors of K'^* are equivalent to those obtained from K' as the variables are linearly dependent. The equations may nevertheless be simpler for instance if the variables β are replaced by normalized variables β^* such that $\int u^{*T} u^* \, dm = \beta^{*T} \beta^*$. The inertia matrix becomes a diagonal matrix which simplifies the simulation.

For a principal axes frame, the matrix $J^* = A^T J A$ has to be diagonal and the constraints can be written as

$$(I_3 - I_2)\theta'_1 = \Lambda_2{}^T \beta,$$
$$(I_1 - I_3)\theta'_2 = \Lambda_1{}^T \beta,$$
$$(I_2 - I_1)\theta'_3 = \Lambda_0{}^T \beta$$

and matrices C_0 and C are then obtained. If only the 3-axis has to be principal, the last constraint disappears and can then be replaced by $\theta'_3 = 0$.

Applications

In a previous paper,[24] we considered the dynamics of a system of point connected rigid bodies in an inverse square field. The equations of motion were written in a generalized mean reference frame. Gravitational terms were explicitly derived and the corresponding linear terms were incorporated in the K matrix. These equations have the appropriate form for stability analysis. As the system is not freely spinning the function V can be taken as testing function and when the damping is complete, necessary and sufficient stability criteria are provided by the matrix K. In this case, the completeness of the damping can easily be checked by Müller's method.[25]

To investigate the attitude stability of a corresponding freely spinning system gravitational terms have to be discarded (they can be recognized by inspection), and the function V' must be considered. In this case the completeness of the damping can be checked by a rather simple extension of the Müller theorem.[26] The application is straightforward, and the equation will not be further developed. Similarly, the expression of the equation in other reference frames is very simple.

The formalism can be extended to a system of interconnected deformable bodies[18]—this constitutes an application of this method to systems with continua. In this case the variables are the angles and displacements in the connections and deformation variables obtained by the finite element method for each body.

The synthesis of the equilibrium configuration is obtained by simultaneously adjusting "large" parameters to satisfy the equilibrium conditions corresponding to the deformations of the joints and determining a "static" large deformation equilibrium under equivalent centrifugal forces for each of the bodies. Once the equilibrium is defined, the various parameters of the system can be determined. It should be noted that the finite elements used to determine these parameters may be different from those used to synthesize the equilibrium, as here a linear elastic model is sufficient.

The explicit derivation of these various matrices is place consuming and has no interest as such. It suffices to say we developed a complete computer program that is able, in its actual version, to investigate the stability of a system of interconnected deformable bodies with up to six degrees of freedom joints. The program requires the knowledge of the equilibrium state and does not allow for prestresses in the structure. In the presence of such stresses, the desired form of the equation is obtained when linearized equations and equilibrium conditions are used simultaneously. This does not present any major theoretical difficulty but almost doubles the size of the computer program, as the symmetry can not be ensured from the beginning but can only be used as a final check.

As an example, we investigated the stability of the European Space Research Organization GEOS satellite. For this analysis, the system was modelled as a main rigid body with two attached flexible cables ended by point masses corresponding to instruments. For this system, fourteen equilibrium configurations could exist.[27] The stability of these configurations was investigated, and we reached the interesting conclusion that only the derived (nominal) equilibrium state is stable.

To check the validity of the results, a modal analysis was performed on this system with rigidified cables and the eigenmodes were taken as initial conditions for a standard simulation program derived from the Roberson-Wittenburg formalism.[13,17] Over one period (computed from the eigenfrequencies) the errors were of the order of 0.01% of the amplitude for all the modes, thus proving the efficiency of both programs. A better accuracy could hardly be expected, the precision on the values of the various variables at equilibrium being of the order of 10^{-5}.

Conclusions

We have derived linearized equations for a spinning deformable system by use of the Lagrange formalism in generalized and/or quasi-coordinates and obtained the corresponding stability criteria. For freely spinning systems, necessary and sufficient conditions have been obtained by using a combination of the Hamiltonian and integrals of motion as a testing function. The first two conditions correspond to the maximum axis of inertia requirement and the other conditions depend on the way the system deforms. These conditions are somewhat simplified when a particular reference frame is chosen. The motion of such a frame has been obtained, and its choice depends on the particular application. The simplest stability criteria are obtained with a principal axis reference frame, and the simulation is simplified when a mean reference frame is used.

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Magnetic Attitude Control System for Dual-Spin Satellites

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A closed-loop control law is developed for a dual-spin satellite control system which utilizes the interaction of the geomagnetic field with the satellite dipole parallel to the spin axis. The control law consists of the linear combination of the pitch axis component of the rate of change of the geomagnetic field and the product of the roll angle and roll axis component of the geomagnetic field. Application of the method of multiple time scales yields approximate solutions for the feedback gains in terms of the system parameters. Approximate solutions are also obtained for the response of the system to disturbance torques. A comparison of the approximate solutions and numerical solutions obtained by numerical integration of the exact equations of motion is then given.

Introduction

INTERACTION between onboard magnets and the geomagnetic field has been used as a means of satellite attitude control.[1–7] It has been used extensively as a method for despinning satellites, fine attitude control,[1–5] and recently has been suggested as a means for acquisition for tumbling satellites.[6–7] The development of a suitable control law for the satellite dipole has been the main concern of the previous investigations. Renard[2] was concerned with obtaining the control law for attitude control which would be activated by ground command. The results of his study showed that the quarter-orbit bang-bang control was the best for near magnetic polar orbits. Wheeler[4] applied the method of averaging to obtain the control law for a closed-loop control system. Shigera[5] developed an on-off control law for a single spin satellite which required switching four times per orbit but not every quarter orbit as did the control system developed by Renard.[2]

This investigation is concerned with the development of the control law governing the strength of a satellite dipole parallel to the spin axis. The interaction of this dipole with the geomagnetic field will provide the attitude control of a dual spin satellite. The control system is to be closed-loop as compared to the open-loop system of Renard.[2] The available data from sensors for development of the control law are the roll angle from the horizon sensor and the geomagnetic field components obtained from the magnetometer. As will be shown later $\dot{B}_\theta$, the rate of change of the geomagnetic field along the pitch axis, is proportional to the yaw and roll rates for small angles. The control law, which was proposed by R. Z. Fowler, President of Ithaco, Inc., is

$$M_\theta = K_1 B_\phi \phi - K_2 \dot{B}_\theta \tag{1}$$

where B_ϕ is the component of the geomagnetic field along the roll axis, ϕ is the roll angle, M_θ is the strength of the pitch magnet, and K_1 and K_2 are the feedback gains. The equations of motion for small angles are linear but the coefficients are time

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